

RENORMALIZATION AND SIEGEL DISKS FOR COMPLEX HÉNON MAPS

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ABSTRACT. We use hyperbolicity of golden-mean renormalization of dissipative Hénon-like maps to prove that the boundaries of Siegel disks of sufficiently dissipative quadratic complex Hénon maps with golden-mean rotation number are topological circles.

Conditionally on an appropriate renormalization hyperbolicity property, we derive the same result for Siegel disks of Hénon maps with all eventually periodic rotation numbers.

1. INTRODUCTION

Consider the complex quadratic Hénon map written as

$$H_{c,a}(x, y) = (x^2 + c + ay, ax) \text{ for } a \neq 0.$$

The maps $H_{c,a}$ and $H_{c,-a}$ are conjugate by the change of coordinates $(x, y) \mapsto (x, -y)$; and the pair of parameters (c, a^2) determines the Hénon map uniquely up to a biholomorphic conjugacy. In this parametrization the Jacobian is $-a^2$. Let K^\pm be the sets of points that do not escape to infinity under forward, respectively backward iterations of the Hénon map. Their topological boundaries are $J^\pm = \partial K^\pm$. Let $K = K^+ \cap K^-$ and $J = J^- \cap J^+$. The sets J^\pm , K^\pm are unbounded, connected sets in \mathbb{C}^2 (see [BS1]). The sets J and K are compact (see [HOV1]). In analogy to one-dimensional dynamics, the set J is called the Julia set of the Hénon map.

In this paper we will always assume that the Hénon map is dissipative, $|a| < 1$. Note that for $a = 0$, the map $H_{c,a}$ degenerates to

$$(x, y) \mapsto (f_c(x), 0),$$

where $f_c(x) = x^2 + c$ is a one-dimensional quadratic polynomial. Thus for a fixed small value of a_0 , the one parameter family H_{c,a_0} is a small perturbation of the quadratic family.

Note that a Hénon map $H_{c,a}$ is determined by the multipliers λ and μ at a fixed point uniquely up to changing the sign of a . In particular,

$$\lambda\mu = -a^2,$$

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the parameter c is a function of a^2 and λ :

$$c = (1 - a^2) \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda} \right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda} \right)^2.$$

Hence, we sometimes write $H_{\lambda,\mu}$ instead of $H_{c,a}$, when convenient. When $\mu = 0$, the Hénon map degenerates to

$$H_{\lambda,0}(x, y) = (P_\lambda(x), 0), \text{ where } P_\lambda(x) = x^2 + \lambda/2 - \lambda^2/4. \quad (1)$$

We say that a dissipative Hénon map $H_{c,a}$ has a *semi-Siegel fixed point* (or simply that $H_{c,a}$ is semi-Siegel) if the eigenvalues of the linear part of $H_{c,a}$ at that fixed point are $\lambda = e^{2\pi i\theta}$, with $\theta \in (0, 1) \setminus \mathbb{Q}$ and μ , with $|\mu| < 1$, and $H_{c,a}$ is locally biholomorphically conjugate to the linear map

$$L(x, y) = (\lambda x, \mu y).$$

The classic theorem of Siegel [Sie] states, in particular, that $H_{\lambda,\mu}$ is semi-Siegel whenever θ is Diophantine, that is $q_{n+1} < cq_n^d$, where p_n/q_n are the continued fraction convergents of θ . The existence of a linearization is a local result, however, in this case there exists a linearizing biholomorphism $\phi : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}^2$ sending $(0, 0)$ to the semi-Siegel fixed point,

$$H_{\lambda,\mu} \circ \phi = \phi \circ L,$$

such that the image $\phi(\mathbb{D} \times \mathbb{C})$ is *maximal* (see [MNTU]). We call $\phi(\mathbb{D} \times \mathbb{C})$ the *Siegel cylinder*; it is a connected component of the interior of K^+ and its boundary coincides with J^+ (see [BS2]). We let

$$\Delta = \phi(\mathbb{D} \times \{0\}),$$

and by analogy with the one-dimensional case call it the *Siegel disk* of the Hénon map. Clearly, the Siegel cylinder is equal to the stable manifold $W^s(\Delta)$, and $\Delta \subset K$ (which is always bounded). Moreover, $\partial\Delta \subset J$, the Julia set of the Hénon map.

Remark 1.1. *Let \mathbf{q} be the semi-Siegel fixed point of the Hénon map. Then $\Delta \subset W^c(\mathbf{q})$, the center manifold of \mathbf{q} (see e.g. [S] for a definition of W^c). The center manifold is not unique in general, but all center manifolds $W^c(\mathbf{q})$ coincide on the Siegel disk. This phenomenon is nicely illustrated in [O], Figure 5.*

The main result of this paper is the following theorem:

Theorem A. *There exists $\delta > 0$ such that the following holds. Let $\theta_* = (\sqrt{5}-1)/2$ be the inverse golden mean, $\lambda_* = e^{2\pi i\theta_*}$, and let $|\mu| < \delta$. Then the boundary of the Siegel disk of $H_{\lambda_*,\mu}$ is a homeomorphic image of the circle.*

By Carathéodory Theorem, the linearizing map

$$\phi : \mathbb{D} \times \{0\} \rightarrow \Delta \quad (2)$$

extends continuously and injectively to the boundary. However, we note:

Theorem B. *The conjugacy*

$$\phi : S^1 \times \{0\} \rightarrow \partial\Delta$$

is not C^1 -smooth.

It is worthwhile mentioning that if we assume that $\lambda = e^{2\pi i\theta}$, $\mu = e^{2\pi i\theta'}$ and the pair (θ, θ') satisfies the two-dimensional Brjuno condition [Brj], then the conservative Hénon map $H_{\lambda, \mu}$ has a bounded maximal domain of linearization, called a *Siegel ball*. Herman [He] asked *whether the boundary of the Siegel ball is a topological or perhaps a C^∞ submanifold of \mathbb{C}^2* . We answer similar questions, in the dissipative setting, as outlined above.

The proofs of Theorems A and B are based on a renormalization theory for two-dimensional dissipative Hénon-like maps, developed by the first and third authors in [GaYa2]. A Hénon-like map (see [dCLM]) $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ can be defined as $H(x, y) = (f(x) + \epsilon(x, y), ax)$, for some small ϵ . In this normalization, it has Jacobian $-a\partial\epsilon/\partial y$ and it reduces to the standard Hénon map when $f(x) = x^2 + c$ and $\epsilon(x, y) = ay$. In general, the Jacobian of a Hénon-like map is not constant. Following [LRT], we say that a Hénon-like map H has a semi-Siegel fixed point if there exists a local holomorphic change of variables ϕ such that $\tilde{H} = \phi \circ H \circ \phi^{-1}$ is a skew product of the form $\tilde{H}(x, y) = (\lambda x, \mu(x)y)$, for some holomorphic function $\mu(x) = \mu + O(x)$, where $\lambda = e^{2\pi i\theta}$, with $\theta \in (0, 1) \setminus \mathbb{Q}$, and $|\mu| < 1$. This condition is equivalent to the existence of a one-dimensional Siegel disk $\Delta = \phi(\mathbb{D} \times \{0\})$.

Below, we will be using several different renormalization operators. The first of them is the renormalization of pairs of two-dimensional dissipative maps introduced in [GaYa2]. We will recall its definition in § 3.

In one complex dimension, it corresponds to the renormalization of *commuting pairs* \mathcal{R} (cf. [Stir]). In particular, suppose that P_λ , defined in (1), has a Siegel disk at the origin, $\lambda = e^{2\pi i\theta}$, and let p_n/q_n denote the n -th continued fraction convergent of θ . Then the n -th renormalization $\mathcal{R}^n P_\lambda$ is a linear (if n is even) or anti-linear (if n is odd) rescaling of the restriction of the pair of iterates $(P_\lambda^{q_n}, P_\lambda^{q_{n+1}})$ to appropriate neighborhoods of the critical point $c_\lambda = 0$.

A different take on renormalization of one-dimensional analytic maps with Siegel disks was introduced by the third author in [Ya1] based on the *cylinder renormalization operator* \mathcal{R}_{cyl} . This operator acts on analytic maps defined in some neighborhood of the origin, rather than on pairs. For a one-dimensional cylinder-renormalizable analytic map f with a Siegel disk with rotation number θ , denote $g = \mathcal{R}_{cyl} f$. There exist constants $s \in \mathbb{N}$ and $K > 1$ such that the following holds. There exists a restriction ζ of the pair (g, g^{q_1}) to a neighborhood of the critical point of f , and a conformal map ψ with K -bounded distortion such that

$$\psi \circ \zeta \circ \psi^{-1} = \mathcal{R}^s(f). \quad (3)$$

For a topological disk $Z \ni 0$ denote $\mathcal{H}(Z)$ the Banach space of holomorphic functions f in Z with the uniform norm, and set $\mathcal{H}(Z, W) \equiv \mathcal{H}(Z) \times \mathcal{H}(W)$. We will typically use the notation (η, ξ) for an element of $\mathcal{H}(Z, W)$.

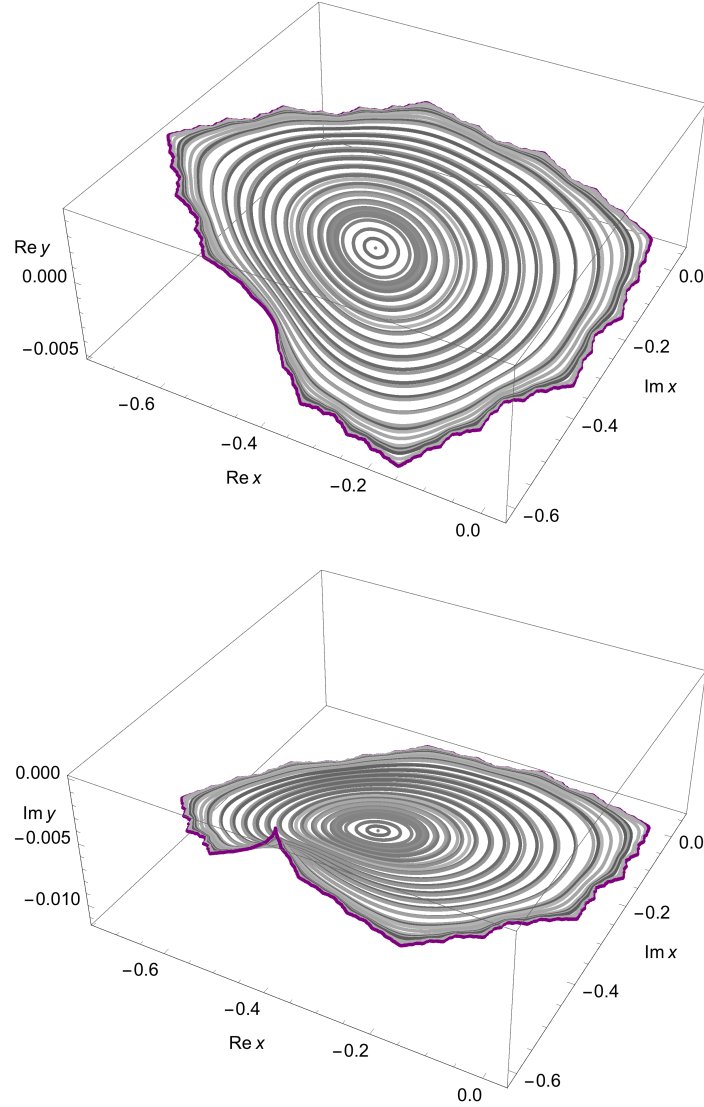


Figure 1. A three dimensional plot of the Siegel disk and its boundary for a Hénon map with a semi-Siegel fixed point with the golden mean rotation number. The parameter $a = 0.01 + 0.01i$. The three axes are as follows: TOP: $\text{Re}(x)$, $\text{Im}(x)$ and $\text{Re}(y)$; BOTTOM: $\text{Re}(x)$, $\text{Im}(x)$ and $\text{Im}(y)$.

We let $\mathcal{C}(Z, W)$ denote the Banach subspace of $\mathcal{H}(Z, W)$ given by the linear conditions

$$\eta'(0) = \xi'(0) = 0.$$

We say that a pair $(\eta, \xi) \in \mathcal{C}(Z, W)$ is *almost commuting to order $s \geq 0$* if the following holds:

$$(\eta \circ \xi)^{(n)}(0) = (\xi \circ \eta)^{(n)}(0), \quad 0 \leq n \leq s; \quad \eta''(0) > 0; \quad \xi''(0) > 0, \quad \text{and } \xi(0) = 1. \quad (4)$$

In the case $s = 2$, we will simply call the pair *almost commuting (or a.c.)*. We denote $\mathcal{B}(Z, W)$ the subset of $\mathcal{C}(Z, W)$ consisting of a.c. pairs. In [GaYa2], it is shown that there exists an open neighborhood \mathcal{U} of $\mathcal{C}(Z, W)$ such that $\mathcal{B}(Z, W) \cap \mathcal{U}$ is a Banach submanifold of $\mathcal{H}(Z, W)$.

Let

$$G(x) = \left\{ \frac{1}{x} \right\}$$

denote the Gauss map. Let θ be periodic under the Gauss map with period p . McMullen in [Mc] showed that there exists a pair of analytic maps ζ_λ which is periodic under the action of \mathcal{R} with period p , and such that for every $\lambda_1 = e^{2\pi i \theta_1}$ where

$$G^m(\theta_1) = \theta, \quad \text{for some } m \geq 0,$$

we have

$$\mathcal{R}^{np+m} P_{\lambda_1} \rightarrow \zeta_\lambda \text{ at a rate, which is geometric in } n.$$

Let θ and p be as above. Set

$$k = p \text{ if } p \text{ is even, or } k = 2p \text{ if } p \text{ is odd.} \quad (5)$$

Let us say that *renormalization hyperbolicity property (H)* holds for θ if the following is true:

(H) *There exist a pair of topological disks $\tilde{Z} \ni Z$, $\tilde{W} \ni W$ and $n = mk$, where $m \in \mathbb{N}$ and k is as in (5) such that*

- (i) *The operator \mathcal{R}^n is an analytic operator from an open neighborhood of its fixed point ζ_λ in $\mathcal{B}(Z, W)$ to $\mathcal{B}(\tilde{Z}, \tilde{W})$.*
- (ii) *The differential $D\mathcal{R}^n|_{\zeta_\lambda}$ is a compact linear operator in $T_{\zeta_\lambda}\mathcal{B}(Z, W)$. Let $M \equiv D\mathcal{R}^n|_{\zeta_\lambda}$. Then M has a single simple eigenvalue outside of the closed unit disk, and the rest of the spectrum of M lies inside the open unit disk.*

We prove a conditional theorem:

Theorem C. *Suppose renormalization hyperbolicity property (H) holds for θ , and let θ_1 be such that $G^m(\theta_1) = \theta$ for some $m \in \mathbb{N}$. Set $\lambda_1 = e^{2\pi i \theta_1}$. Then the following statements hold:*

- (I) *there exists $\delta > 0$ such that if $|\mu| < \delta$ then the map $H_{\lambda_1, \mu}$ lies in the stable set of ζ_λ ;*
- (II) *every Hénon-like map H in $W^s(\zeta_\lambda)$ has a Siegel disk Δ_H whose boundary is a topological circle;*
- (III) *the Carathéodory extension of the linearizing coordinate ϕ as in equation (2) to a map $S^1 \times \{0\} \rightarrow \partial\Delta_H$ is not C^1 -smooth.*

Our Theorems A and B will follow from Theorem C and the following statement proven in [GaYa2]:

Golden-mean renormalization hyperbolicity [GaYa2]. *Renormalization hyperbolicity property (H) holds for $\theta_* = (\sqrt{5} - 1)/2$.*

2. DYNAMICAL PARTITIONS AND MULTI-INDICES

Consider the space \mathcal{I} of multi-indices $\bar{s} = (a_1, b_1, a_2, b_2, \dots, a_n, b_n)$ where $a_j \in \mathbb{N}$ for $2 \leq n$, $a_1 \in \mathbb{N} \cup \{0\}$, $b_j \in \mathbb{N}$ for $1 \leq j \leq n-1$, and $b_n \in \mathbb{N} \cup \{0\}$. We introduce a partial ordering on multi-indices: $\bar{s} \succ \bar{t}$ if $\bar{s} = (a_1, b_1, a_2, b_2, \dots, a_n, b_n)$, $\bar{t} = (a_1, b_1, \dots, a_k, b_k, c, d)$, where $k < n$ and either $c < a_{k+1}$ and $d = 0$ or $c = a_{k+1}$ and $d < b_{k+1}$. For such a pair, we also define

$$\bar{q} \equiv \bar{s} \ominus \bar{t} :$$

- in the case when $d = 0$, $\bar{q} = (a_{k+1} - c, b_{k+1}, \dots, a_n, b_n)$;
- in the other case, $\bar{q} = (0, b_{k+1} - d, a_{k+2}, b_{k+2}, \dots, a_n, b_n)$.

For a pair of maps $\zeta = (\eta, \xi)$ and \bar{s} as above we will denote

$$\zeta^{\bar{s}} \equiv \xi^{b_n} \circ \eta^{a_n} \circ \dots \circ \xi^{b_2} \circ \eta^{a_2} \circ \xi^{b_1} \circ \eta^{a_1}.$$

Similarly,

$$\zeta^{-\bar{s}} \equiv (\zeta^{\bar{s}})^{-1} = (\eta^{a_1})^{-1} \circ (\xi^{b_1})^{-1} \circ \dots \circ (\eta^{a_n})^{-1} \circ (\xi^{b_n})^{-1}.$$

Consider the n -th pre-renormalization of ζ :

$$p\mathcal{R}^n \zeta = \zeta_n = (\eta_n|_{Z_n}, \xi_n|_{W_n}),$$

where $Z_n = \alpha_n(Z)$, $W_n = \alpha_n(W)$, and

$$\alpha_n(z) = \eta_n(0)z. \tag{6}$$

We define $\bar{s}_n, \bar{t}_n \in \mathcal{I}$ to be such that

$$\eta_n = \zeta^{\bar{s}_n}, \text{ and } \xi_n = \zeta^{\bar{t}_n}.$$

Let $T_\theta : \mathbb{R} \rightarrow \mathbb{R}$ be the translation $x \mapsto x + \theta$, with $\lambda = \exp(2\pi i\theta)$. Define

$$f(x) = T_{\theta_1}^2(x) - 1 \text{ and } g(x) = T_{\theta_1}(x) - 1,$$

and set

$$I = [g(0), 0], \quad J = [0, f(0)], \text{ and } H = (f|_I, g|_J).$$

Define $H_n = (f_n, g_n) = (H^{\bar{s}_n}, H^{\bar{t}_n})$, and set

$$I_n = [0, g_n(0)], \quad J_n = [0, f_n(0)].$$

Now consider the collection of intervals

$$\mathcal{P}_n \equiv \{H^{\bar{w}}(I_n) \text{ for all } \bar{w} \prec \bar{s}_n \text{ and } H^{\bar{w}}(J_n) \text{ for all } \bar{w} \prec \bar{t}_n\}. \tag{7}$$

It is easy to see that:

$$(a) \quad \bigcup_{T \in \mathcal{P}_n} T = I \cup J;$$

- (b) for any two distinct elements T_1 and T_2 of \mathcal{P}_n , the interiors of T_1 and T_2 are disjoint.

In view of the above, we call \mathcal{P}_n the n -th dynamical partition of the segment $I \cup J$.

Consider the sequence of domains

$$\mathcal{V}_n \equiv \{\zeta^{\bar{w}}(Z_n) \text{ for all } \bar{w} \prec \bar{s}_n \text{ and } \zeta^{\bar{w}}(W_n) \text{ for all } \bar{w} \prec \bar{t}_n\}.$$

By analogy with the above definition (and somewhat abusing the notation) we call \mathcal{V}_n the n -th dynamical partition of the pair ζ .

Proposition 2.1. *Suppose, renormalization hyperbolicity property holds for θ , and*

$$\zeta \in W^s(\zeta_\lambda), \text{ where } \lambda = e^{2\pi i \theta}.$$

Then there exists $N = N(\zeta)$, $K > 0$, and $0 < \gamma < 1$ so that for every $n > N$ the following properties hold.

- 1) *If $Q_n \in \mathcal{V}_n$ then $\text{diam}(Q_n) < \gamma^n$.*
- 2) *Any two neighboring domains $Q_n, Q'_n \in \mathcal{V}_n$ are K -commensurable.*
- 3) *For every $\bar{w} \prec \bar{s}_n$ (or $\bar{w} \prec \bar{t}_n$) set $\psi_{\bar{w}}^\zeta = \zeta^{\bar{w}}\alpha_n$. Then $\|D\psi_{\bar{w}}^\zeta|_Z\|_\infty < \gamma^n$ (or $\|D\psi_{\bar{w}}^\zeta|_W\|_\infty < \gamma^n$, respectively).*

Proof. By our assumption, there exists $N > 0$ and a pair of domains $\hat{Z} \ni Z$ and $\hat{W} \ni W$ such that for all $n \geq N$ the maps of the pair $\mathcal{R}^n \zeta \in \mathcal{C}(\hat{Z}, \hat{W})$. By Koebe Distortion Theorem, this implies that for all $\bar{w} \prec \bar{s}_n$ (or $\bar{w} \prec \bar{t}_n$) the branches $\zeta^{-\bar{w}}$ have bounded distortion. The claims readily follow. \square

3. RENORMALIZATION FOR PAIRS OF TWO-DIMENSIONAL DISSIPATIVE MAPS

This Section contains a summary of the extention of the renormalization operator from the space $\mathcal{B}(Z, W)$ of almost commuting pairs to an appropriately defined space of two-dimensional maps. The details of the procedure can be found in [GaYa3].

Let Ω, Γ be domains in \mathbb{C}^2 . We denote $O(\Omega, \Gamma)$ the Banach space of bounded analytic functions $F = (F_1(x, y), F_2(x, y))$ from Ω and Γ respectively to \mathbb{C}^2 equipped with the norm

$$\|F\| = \frac{1}{2} \left(\sup_{(x,y) \in \Omega} |F_1(x, y)| + \sup_{(x,y) \in \Gamma} |F_2(x, y)| \right). \quad (8)$$

We let $O(\Omega, \Gamma, \delta)$ stand for the δ -ball around the origin in this Banach space.

In what follows, we fix W, Z, \tilde{Z} , and \tilde{W} as in **(H)**, and $R > 0$ such that $\mathbb{D}_R \subset Z \cap W$, and let $\Omega = Z \times \mathbb{D}_R, \Gamma = W \times \mathbb{D}_R$. We select \hat{Z} and \hat{W} so that

$$Z \Subset \hat{Z} \Subset \tilde{Z}, \quad W \Subset \hat{W} \Subset \tilde{W}.$$

We define an isometric embedding ι of the space $\mathcal{H}(Z, W)$ into $O(\Omega, \Gamma)$ which send the pair $\zeta = (\eta, \xi)$ to the pair of functions $\iota(\zeta)$:

$$\left(\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \eta(x) \\ \eta(y) \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \xi(x) \\ \xi(y) \end{pmatrix} \right). \quad (9)$$

Let \mathcal{U} be an open neighborhood of ζ_λ in $\mathcal{C}(Z, W)$, and let Q be a neighborhood of 0 in \mathbb{C} . We will consider an open subset of $O(\Omega, \Gamma)$ of pairs of maps of the form

$$A(x, y) = (a(x, y), h(x, y)) = (a_y(x), h_y(x)), \quad (10)$$

$$B(x, y) = (b(x, y), g(x, y)) = (b_y(x), g_y(x)), \quad (11)$$

such that

- 1) the pair $(a(x, y), b(x, y))$ is in a δ -neighborhood of \mathcal{U} in $O(\Omega, \Gamma)$,
- 2) $(h, g) \in O(\Omega, \Gamma)$ are such that $|\partial_x h(x, 0)| > 0$ and $|\partial_x g(x, 0)| > 0$ whenever $x \notin \bar{Q}$, and

$$(h(x, y) - h(x, 0), g(x, y) - g(x, 0)) \in O(\Omega, \Gamma, \delta).$$

This open subset of $O(\Omega, \Gamma)$ will be denoted $\mathcal{A}(\mathcal{U}, Q, \delta)$ for brevity.

We say that a pair (A, B) is a renormalization of a map H if

$$(A = H^{q_n}, B = H^{q_{n+1}}) \text{ for some } n \geq 0.$$

Given a pair $\Sigma = (A, B)$ as in (10) – (11) we set

$$\mathcal{L}\Sigma \equiv (a(x, 0), b(x, 0)).$$

3.1. Defining renormalization: coordinate transformations. Let $(\eta, \xi) \in \mathcal{B}(Z, W)$ be $n \geq 2$ times renormalizable, and consider its n -th pre-renormalization written as

$$p\mathcal{R}^n \zeta = (\zeta^{\bar{s}_n}, \zeta^{\bar{t}_n}),$$

and denote

$$\begin{aligned} \hat{s}_n &= \begin{cases} (a_1, b_1, a_2, b_2, \dots, a_n - 2, b_n), & a_n \geq 2 \\ (a_1, b_1, a_2, b_2, \dots, 0, b_n - 1), & a_n = 1 \end{cases}, \\ \phi_0(x) &= \begin{cases} \eta^2, & a_n \geq 2 \\ \eta \circ \xi, & a_n = 1 \end{cases}. \end{aligned}$$

Define \hat{t}_n in a similar way. Then $p\mathcal{R}^n \zeta$ can be written as

$$p\mathcal{R}^n \zeta = \phi_0 \circ (\zeta^{\hat{s}_n}, \zeta^{\hat{t}_n}).$$

For a sufficiently large n , the function η^{-1} is a diffeomorphism of the neighborhood $\alpha_n(Z \cup W)$, and one can define the n -th pre-renormalization of ζ in $\eta^{-1}(\alpha_n(Z \cup W))$ as

$$\hat{p}\mathcal{R}^n \zeta = (\eta^{-1} \circ \zeta^{\bar{s}_n} \circ \eta, \eta^{-1} \circ \zeta^{\bar{t}_n} \circ \eta) = (f \circ \zeta^{\hat{s}_n} \circ \eta, f \circ \zeta^{\hat{t}_n} \circ \eta),$$

where $f = \eta$ if $a_n \geq 2$ and $f = \xi$ if $a_n = 1$.

Next, suppose that $\Sigma = (A, B)$ lies in $\mathcal{A}(\mathcal{U}, Q, \delta)$ with \mathcal{U} and δ sufficiently small, so that the following pre-renormalization is defined in a neighborhood of $\eta^{-1}(\alpha_n(Z \cup W)) \times \{0\}$:

$$\hat{p}\mathcal{R}^n\Sigma = \left(F \circ \Sigma^{\hat{s}_n} \circ A, F \circ \Sigma^{\hat{t}_n} \circ A \right),$$

where $F = A$ if $a_n \geq 2$ and $F = B$ if $a_n = 1$.

We will denote

$$\pi_1(x, y) = x \text{ and } \pi_2(x, y) = y.$$

Set

$$\phi_y(x) = \phi(x, y) := \begin{cases} \pi_1 A^2(x, y), & a_n \geq 2 \\ \pi_1 A \circ B(x, y), & a_n = 1 \end{cases}$$

For sufficiently small δ , the map ϕ_z is close to ϕ_0 and is a diffeomorphism of a neighborhood of $\pi_1 \Sigma^{\hat{s}_n}(\alpha_n(Z), 0) \approx \zeta^{\hat{s}_n}(\alpha_n(Z))$ for all $z \in \mathbb{D}_R$ for some $R = R(\delta) > 0$. Similarly, g_z is a diffeomorphism of a neighborhood of $\pi_1 \Sigma^{\hat{s}_n}(\alpha_n(Z), 0)$ for all $z \in \mathbb{D}_R$ for some $R = R(\delta) > 0$.

Furthermore, set

$$q_z(x) \equiv q(x, z) = \pi_2 F(x, z) = \begin{cases} h_z(x), & a_n \geq 2 \\ g_z(x), & a_n = 1 \end{cases}$$

According to our definition of the class $\mathcal{A}(\mathcal{U}, Q, \delta)$, this is a diffeomorphism outside a neighborhood of zero. Also, set

$$w_z(x) \equiv w(x, z) := q_z(\phi_z^{-1}(x)),$$

a diffeomorphism of a neighborhood of $\pi_1 \phi_z \circ \Sigma^{\hat{s}_n}(\alpha_n(Z), 0)$ in \mathbb{C}^2 onto its image for all $z \in \mathbb{D}_R$ for some $R = R(\delta) > 0$. Notice, that $\partial_z w_z(x)$ and $\partial_z w_z^{-1}(x)$ are functions whose uniform norms are $O(\delta)$.

Define the following transformation:

$$H_\Sigma(x, y) = (a_y(x), w_{q_0^{-1}(y)}^{-1}(y)), \quad (12)$$

This transformation is δ -close to $(\eta(x), \phi_0(q_0^{-1}(y)))$ in $O(\Omega, \Gamma)$, and therefore, for small δ , is a diffeomorphism of a neighborhood of $\pi_1 F \circ \Sigma^{\hat{s}_n}(\alpha_n(Z), 0) \approx f(\zeta^{\hat{s}_n}(\alpha_n(Z)))$ onto its image. In particular,

$$A \circ H_\Sigma^{-1}(x, y) = (x, h(\eta^{-1}(x), y)) + O(\epsilon). \quad (13)$$

We use $H_\Sigma(x, y)$ to pull back $\hat{p}\mathcal{R}^n\Sigma$ to a neighborhood of definition of the n -th pre-renormalization of a pair (η, ξ) - that is, a neighborhood of $\alpha_n(Z \cup W)$ in \mathbb{C}^2 :

$$p\mathcal{R}^n\Sigma = (\bar{A}, \bar{B}) = H_\Sigma \circ F \circ \left(\Sigma^{\hat{s}_n}, \Sigma^{\hat{t}_n} \right) \circ A \circ H_\Sigma^{-1}(x, y).$$

The following has been proved in [GaYa2].

Lemma 3.1. *There exists an $n \in \mathbb{N}$, and a choice of \mathcal{U} , Q , δ_0 and $C > 0$ such that the following holds. For every $\delta < \delta_0$ and every $\Sigma \in \mathcal{A}(\mathcal{U}, Q, \delta)$ the pair $p\mathcal{R}^n\Sigma$ is defined, lies in $O(\hat{\Omega}, \hat{\Gamma})$, $\hat{\Omega} = \hat{Z} \times \mathbb{D}_R$, $\hat{\Gamma} = \hat{W} \times \mathbb{D}_R$, and*

$$\text{dist}(p\mathcal{R}^n\Sigma, \iota(\mathcal{H}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))) < C\delta(\|\pi_1\Sigma - \pi_2\Sigma\| + \delta).$$

Let us write

$$\bar{A}(x, y) = \begin{pmatrix} \bar{\eta}_1(x) + \bar{\tau}_1(x, y) \\ \bar{\eta}_2(x) + \bar{\tau}_2(x, y) \end{pmatrix}, \quad (14)$$

where

$$\bar{\eta}_1(x) \equiv \pi_1\bar{A}(x, 0), \quad \bar{\eta}_2(x) \equiv \pi_2\bar{A}(x, 0)$$

are $O(\delta\|\pi_1\Sigma - \pi_2\Sigma\| + \delta^2)$ -close to each other, and both are δ -close to $\pi_\eta p\mathcal{R}^n\zeta = \zeta^{\bar{s}_n}$, where π_η and π_ξ are the projections on, correspondingly, the first and the second map in a pair, and

$$\bar{\tau}_1(x, y) \equiv \pi_1\bar{A}(x, y) - \pi_1\bar{A}(x, 0), \quad \bar{\tau}_2(x, y) \equiv \pi_2\bar{A}(x, y) - \pi_2\bar{A}(x, 0),$$

are functions whose norms are $O(\delta^2)$. Similarly,

$$\bar{B}(x, y) = \begin{pmatrix} \bar{\xi}_1(x) + \bar{\pi}_1(x, y) \\ \bar{\xi}_2(x) + \bar{\pi}_2(x, y) \end{pmatrix},$$

where

$$\bar{\xi}_1(x) \equiv \pi_1\bar{B}(x, 0), \quad \bar{\xi}_2(x) \equiv \pi_2\bar{B}(x, 0)$$

are $O(\delta\|\pi_1\Sigma - \pi_2\Sigma\| + \delta^2)$ -close to each other, and both are δ -close to $\pi_\xi p\mathcal{R}^n\zeta = \zeta^{\bar{t}_n}$, and

$$\bar{\pi}_1(x, y) \equiv \pi_1\bar{B}(x, y) - \pi_1\bar{B}(x, 0), \quad \bar{\pi}_2(x, y) \equiv \pi_2\bar{B}(x, y) - \pi_2\bar{B}(x, 0),$$

are functions whose norms are $O(\delta^2)$.

3.2. Defining renormalization: critical projection. By the Argument Principle, if δ is sufficiently small, then the function $\pi_1\bar{B} \circ \bar{A}(x, 0)$ has a unique critical point c_1 in a neighborhood of 0. Set $T_1(x, y) = (x + c_1, y)$, then

$$\partial_x (\pi_1 T_1^{-1} \circ \bar{B} \circ \bar{A} \circ T_1)(0, 0) = 0.$$

Similarly, if δ is sufficiently small, the function $\pi_1 T_1^{-1} \circ \bar{A} \circ \bar{B} \circ T_1(x, 0)$ has a unique critical point c_2 in a neighborhood of 0. Set $T_2(x, y) = (x + c_2, y)$, then

$$\partial_x (\pi_1 T_2^{-1} \circ T_1^{-1} \circ \bar{A} \circ \bar{B} \circ T_1 \circ T_2)(0, 0) = 0.$$

We now set

$$\begin{aligned} \Pi_1(\bar{A}, \bar{B}) &= (\tilde{A}, \tilde{B}) := (T_2^{-1} \circ T_1^{-1} \circ \bar{A} \circ T_1, T_1^{-1} \circ \bar{B} \circ T_1 \circ T_2) \\ &= \left(\begin{pmatrix} \tilde{\eta}_1(x) + \tilde{\tau}_1(x, y) \\ \tilde{\eta}_2(x) + \tilde{\tau}_2(x, y) \end{pmatrix}, \begin{pmatrix} \tilde{\xi}_1(x) + \tilde{\pi}_1(x, y) \\ \tilde{\xi}_2(x) + \tilde{\pi}_2(x, y) \end{pmatrix} \right), \end{aligned}$$

where the norms of the functions $\tilde{\tau}_k, \tilde{\pi}_k$, $k = 1, 2$, are $O(\delta^2)$.

The critical points of the functions $\pi_1(\bar{A} \circ \bar{B})(x, 0)$ and $\pi_1(\bar{B} \circ \bar{A})(x, 0)$ are $O(\delta\|\pi_1\Sigma - \pi_2\Sigma\| + \delta^2)$ -close to each other, and therefore,

$$T_2 = \text{Id} + O(\delta\|\pi_1\Sigma - \pi_2\Sigma\| + \delta^2). \quad (15)$$

Let us set

$$\tilde{\Sigma} = (\tilde{A}, \tilde{B}) = \Pi_1 p \mathcal{R}^n \Sigma.$$

We note that if the maps \bar{A} and \bar{B} commute, then the critical point of $\pi_1 T_1^{-1} \circ \bar{A} \circ \bar{B} \circ T_1(x, 0)$ is at 0. We, therefore, have the following

Proposition 3.2. *Suppose (A, B) is a renormalization of a map H . Then the map $T_2 \equiv \text{Id}$, and hence, the projection Π_1 is a conjugacy by T_1 .*

3.3. Defining renormalization: commutation projection. At the next step we will project the pair (\tilde{A}, \tilde{B}) onto the subset of pairs satisfying the following almost commutation conditions:

$$\partial_x^i \pi_1(\tilde{A} \circ \tilde{B}(x, 0) - \tilde{B} \circ \tilde{A}(x, 0))|_{x=0} = 0, \quad i = 0, 2 \quad (16)$$

$$\pi_1 \tilde{B}(0, 0) = 1. \quad (17)$$

To that end we set

$$\Pi_2(\tilde{A}, \tilde{B})(x, y) = \left(\begin{pmatrix} \tilde{\eta}_1(x) + ax^4 + bx^6 + \tilde{\tau}_1(x, y) \\ \tilde{\eta}_2(x) + ax^4 + bx^6 + \tilde{\tau}_2(x, y) \end{pmatrix}, \begin{pmatrix} \tilde{\xi}_1(x) + c + \tilde{\pi}_1(x, y) \\ \tilde{\xi}_2(x) + c + \tilde{\pi}_2(x, y) \end{pmatrix} \right),$$

and require that (16) and (17) are satisfied for maps in the pair $\Pi_2(\tilde{A}, \tilde{B})(x, y)$. The following Proposition is proved in [GaYa2].

Proposition 3.3. *There exists $\rho > 0$ such that for all $\tilde{\Sigma}$ in the ρ -neighborhood of*

$$\iota(\mathcal{C}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))$$

there is a unique tuple (a, b, c, d) such that the pair $\Pi_2(\tilde{A}, \tilde{B})$ satisfies the equations (16) and (17). Moreover, in this neighborhood, the dependence of Π_2 on Σ is analytic. Furthermore, if $A \circ B = B \circ A$, then $\Pi_2 = \text{Id}$.

Let us fix $n \in 2\mathbb{N}$, \mathcal{U} , Q , δ so that Lemma 3.1 holds, and furthermore, the image $\Pi_1 p \mathcal{R}^n \mathcal{A}(\mathcal{U}, Q, \delta)$ lies in the ρ -neighborhood of $\iota(\mathcal{C}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))$ as in Proposition 3.3. We then have:

Proposition 3.4. *For every $\Sigma \in \mathcal{A}(\mathcal{U}, Q, \delta)$,*

$$\text{dist}(\Pi_2 \Pi_1 p \mathcal{R}^n \Sigma, \iota(\mathcal{B}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))) < C\delta(\|\pi_1\Sigma - \pi_2\Sigma\| + \delta).$$

Let $\ell_n = \pi_1 \tilde{B}(0, 0)$ and $\Lambda_n(x, y) = (\ell_n x, \ell_n y)$.

Definition 3.5. We define the n -th renormalization of a pair $\Sigma \in \mathcal{A}(\mathcal{U}, Q, \delta)$ as

$$\mathcal{R}^n \Sigma = \Lambda_n^{-1} \circ \Pi_2 \circ \Pi_1 \circ p \mathcal{R}^n \Sigma \circ \Lambda_n. \quad (18)$$

Given a map H from a subset of \mathbb{C}^2 to \mathbb{C}^2 , such that the pair $(H^{q_{N+1}}, H^{q_N}) \in \mathcal{A}(\mathcal{U}, Q, \delta)$ for some integer N , we will also use the shorthand notation

$$\mathcal{R}^n H \equiv \mathcal{R}^n(H^{q_{N+1}}, H^{q_N}).$$

3.4. Hyperbolicity of renormalization of 2D dissipative maps. Let n be an even number as above. We conclude this section by formulating the following theorem:

Theorem 3.6. *Given a p -periodic θ , set that $\lambda = e^{2\pi i\theta}$. Assume that (H) holds. Then there exists $n = mk$, where $m \in \mathbb{N}$ and k is as in (5), such that the point $\iota(\zeta_\lambda)$ is a fixed point of \mathcal{R}^n in $O(\Omega, \Gamma)$. The linear operator $N = D\mathcal{R}^n|_{\iota(\zeta_\lambda)}$ is compact. The spectrum of N coincides with the spectrum of M , where M is as in (H). More specifically, $\kappa \neq 0$ is an eigenvalue of M , and h is a corresponding eigenvector if and only if κ is an eigenvalue of N , and $D\iota(h)$ is a corresponding eigenvector.*

Proof. Since ι is an immersion on $\mathcal{C}(Z, W)$, and

$$\iota \circ \mathcal{R}^k = \mathcal{R}^k \circ \iota,$$

the spectral decomposition of N splits into the direct sum $T_1 \oplus T_2$, where T_1 is the tangent subspace

$$T_1 = T_{\iota(\zeta_\lambda)}\iota(\mathcal{B}(Z, W)).$$

The restriction $N|_{T_1}$ is isomorphic to M . Further, by Proposition 3.4, the magnitude of a perturbation of $\iota(\zeta_\lambda)$ in the direction of a vector in T_2 is decreased quadratically by $(\mathcal{R}^n)^2$. Hence, in the spectral decomposition, the subspace T_2 corresponds to the zero eigenvalue. \square

4. PROOF OF THEOREM C.

4.1. The Hénon family intersects $W^s(\zeta_\lambda)$. Let us fix $\theta, \theta_1, \lambda, \lambda_1$ as in Theorem C. As before, let k be as in (5), and let n be as in Theorem 3.6. Set

$$\mathcal{R} = \mathcal{R}^n. \tag{19}$$

We prove:

Theorem 4.1. *There exists $\delta > 0$ such that if $|\mu| < \delta$ then the one-parameter family $l \rightarrow H_{l,\mu}$ intersects the stable set of ζ_λ under \mathcal{R} .*

Proof. Let $U \ni 0$ be a Jordan domain in \mathbb{C} and let \mathbf{C}_U denote the Banach space of bounded analytic maps f in U equipped with a uniform norm $\|\cdot\|_U$ and such that $f(0) = 0$. Let f_* be the periodic point of \mathcal{R}_{cyl} with $f'_*(0) = e^{2\pi i\theta}$ constructed in [Ya1]. We denote the period of f_* under \mathcal{R}_{cyl} by p . As shown in [Ya1], there exists a choice of domains $U_1 \ni U$ such that

$$f_* \in \mathbf{C}_U \text{ and } \mathcal{R}_{cyl} f_* \in \mathbf{C}_{U_1}.$$

Let n be as in Theorem 3.6. For the quadratic polynomial P_{λ_1} there exists N such that its Nn -th cylinder renormalization lies in the local stable set of f_* in \mathbf{C}_U .

As is shown in [Ya1], the family $l \mapsto \mathcal{R}_{cyl}^N P_l$ lies in the unstable cone field of \mathcal{R}_{cyl} . Specifically, if

$$l_t = \lambda + t,$$

then

$$\|\mathcal{R}_{cyl}^{(i+N)n} P_{l_t} - \mathcal{R}_{cyl}^{(i+N)n} P_\lambda\|_U = a\beta^i t + o(t), \text{ where } \beta > 1 \text{ and } a > 0. \quad (20)$$

Let us select i large enough, so that $\mathcal{R}_{cyl}^{(i+N)n} P_\lambda \in \mathbf{C}_{U_2}$ with $U_2 \ni U$. By Koebe Distortion Theorem,

$$\|\mathcal{R}_{cyl}^{(i+N)n} P_{l_t} - \mathcal{R}_{cyl}^{(i+N)n} P_\lambda\|_U \sim |\mathcal{R}_{cyl}^{(i+N)n} P_{l_t}(0) - \mathcal{R}_{cyl}^{(i+N)n} P_\lambda(0)|, \quad (21)$$

where 0 is the critical point.

Let us turn to renormalization of commuting pairs. We recall that, according to (3), s steps of \mathcal{R} correspond to n steps of the operator \mathcal{R}_{cyl} . Using Koebe Distortion Theorem again, we see that

$$\|\mathcal{R}^{(i+N)s} P_{l_t} - \mathcal{R}^{(i+N)s} P_\lambda\| \sim |\mathcal{R}^{(i+N)s} P_{l_t}(0) - \mathcal{R}^{(i+N)s} P_\lambda(0)|. \quad (22)$$

Note that $\mathcal{R}^{(i+N)s} P_{l_t}$ and $\mathcal{R}^{(i+N)s} P_\lambda$ are restrictions of $\mathcal{R}_{cyl}^{(i+N)n} P_{l_t}$ and $\mathcal{R}_{cyl}^{(i+N)n} P_\lambda$ respectively, rescaled by changes of coordinate Φ_t^{-1} , Φ_0^{-1} respectively. Φ_t analytically depends on the map (see [GaYa1]), and the same considerations as above imply that

$$\|\Phi_t - \Phi_0\| \sim \|\mathcal{R}^{(i+N)s} P_{l_t} - \mathcal{R}^{(i+N)s} P_\lambda\|. \quad (23)$$

The estimates (20)-(23) imply that

$$\|\mathcal{R}^{(i+N)s} P_{l_t} - \mathcal{R}^{(i+N)s} P_\lambda\| \sim \beta^i.$$

Thus the family

$$l \mapsto g_l \equiv \mathcal{R}^{Ns} P_l$$

lies in the expanding cone field of ζ_λ under \mathcal{R} .

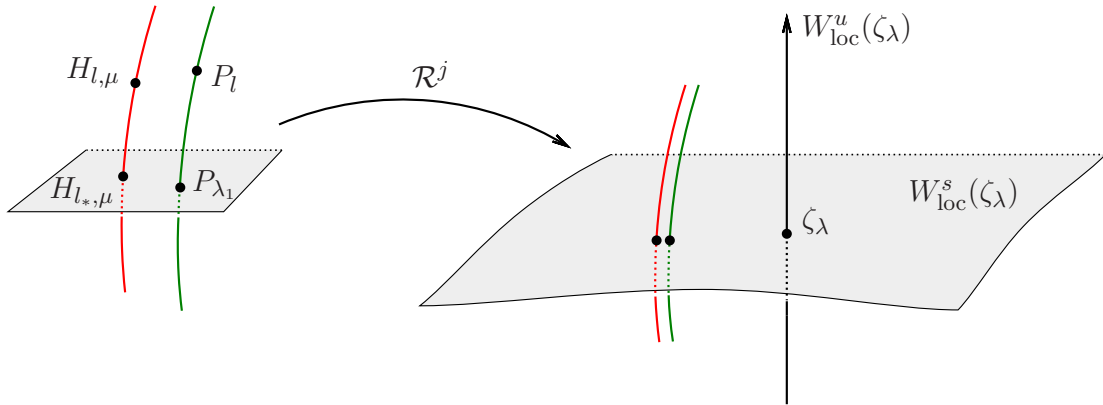


Figure 2. An illustration to the proof of Theorem 4.1; $j = Nns$.

Since for a small enough μ , the family

$$l \mapsto G_l \equiv \mathcal{R}^{N^s} H_{l,\mu}$$

is a C^1 -small perturbation of g_l , it is transverse to $W_{\text{loc}}^s(\zeta_\lambda)$ and hence, intersects with it (see Fig. 2). \square

4.2. Construction of an invariant curve. In this section we prove the following statement:

Proposition 4.2. *There exists $\epsilon > 0$ such that the following holds. Let $|\mu| < \epsilon$, and*

$$H_{l_*,\mu} \in W^s(\zeta_\lambda) \text{ where } \lambda = e^{2\pi i \theta}.$$

Denote Ω_n, Γ_n the domains of definition of the n -th pre-renormalization $p\mathcal{R}^n H_{\lambda_1,\mu}$. Then there exists a curve $\gamma_ \subset \mathbb{C}^2$ such that the following properties hold:*

- γ_* is a homeomorphic image of the circle;
- $\gamma_* \cap \Omega_n \neq \emptyset$ and $\gamma_* \cap \Gamma_n \neq \emptyset$ for all $n \geq 0$;
- there exists a topological conjugacy

$$\varphi_* : \mathbb{T} \rightarrow \gamma_*$$

between the rigid rotation $x \mapsto x + \theta_1 \bmod \mathbb{Z}$ and $H_{l_,\mu}|_{\gamma_*}$;*

- *there exists m such that $G^m(\theta_1) = \theta$;*
- *the conjugacy φ_* is not C^1 -smooth.*

Before proving the above Proposition, we need to introduce some further notation. Below, for brevity, we will denote $\Upsilon^1 = \Omega, \Upsilon^2 = \Gamma$.

We set $n = km$, as in Theorem 3.6 for some $m \geq 1$ (to be fixed later).

To differentiate between transformations for different pairs we will use the following notation. Given a pair Σ , denote Λ_Σ the rescaling that corresponds to the first renormalization \mathcal{R} , and H_Σ - the transformation constructed for Σ in (12), that is

$$\mathcal{R}\Sigma = \Lambda_\Sigma^{-1} \circ H_\Sigma \circ \left(\Sigma^{\tilde{s}_n}, \Sigma^{\tilde{t}_n} \right) \circ H_\Sigma^{-1} \circ \Lambda_\Sigma = L_\Sigma^{-1} \circ \hat{p}\mathcal{R}^n \Sigma \circ L_\Sigma,$$

where

$$\tilde{s}_n = (1, 0, a_1, b_1, \dots, a_n - 1, b_n), \quad (24)$$

(and similarly for \tilde{t}_n), and

$$L_\Sigma = T_\Sigma \circ H_\Sigma^{-1} \circ \Lambda_\Sigma.$$

Note that since the elements of Σ commute, the projection $\Pi_2 = \text{Id}$ and Π_1 is the conjugation by the translation $T_\Sigma := T_1$.

It is instructive to note that $\mathcal{R}^l \Sigma \neq \mathcal{R}^{ln} \Sigma$:

$$\mathcal{R}^l \Sigma = L_{\mathcal{R}^{l-1}\Sigma}^{-1} \circ \dots \circ L_\Sigma^{-1} \circ \hat{p}\mathcal{R}^{ln} \Sigma \circ L_\Sigma \circ \dots \circ L_{\mathcal{R}^{l-1}\Sigma} \neq \Lambda_{l_n}^{-1} \circ H_\Sigma \circ \hat{p}\mathcal{R}^{ln} \Sigma \circ H_\Sigma^{-1} \circ \Lambda_{l_n} = \mathcal{R}^{ln} \Sigma.$$

For each multi-index

$$\bar{w} = (a_0, b_0, a_1, b_1, \dots, a_k, b_k) \prec \tilde{s}_{ln} \text{ or } \bar{w} = (a_1, b_1, \dots, a_k, b_k) \prec \tilde{t}_{ln}$$

we define a domain

$$Q_{\bar{w}}^i = \Sigma^{\bar{w}} \circ L_{\Sigma} \circ L_{\mathcal{R}\Sigma} \circ \dots \circ L_{\mathcal{R}^{l-1}\Sigma}(\Upsilon^i), \quad i = 1 \text{ or } 2. \quad (25)$$

By analogy with a dynamical partition of a commuting pair from Section 2, the collection

$$\mathcal{Q}_{ln} \equiv \{Q_{\bar{w}}^i\}$$

will be referred to as the ln -th partition for the two-dimensional pair Σ .

Given $\Sigma \in W_{\text{loc}}^s(\zeta_\lambda)$, consider the following collection of functions defined on $\Omega \cup \Gamma$:

$$\Psi_{\bar{w}}^{\Sigma} = \Sigma^{\bar{w}} \circ L_{\Sigma}.$$

Given a collection of index sets $\{\bar{w}^i\}$, $\bar{w}^i \prec \bar{s}_n$ or $\bar{w}^i \prec \bar{t}_n$, consider the following *renormalization microscope*

$$\Phi_{\bar{w}^0, \bar{w}^1, \bar{w}^2, \dots, \bar{w}^{j-1}, \Sigma}^j = \Psi_{\bar{w}^0}^{\Sigma} \circ \Psi_{\bar{w}^1}^{\mathcal{R}\Sigma} \circ \dots \circ \Psi_{\bar{w}^{j-1}}^{\mathcal{R}^{(j-1)}\Sigma},$$

which we will also denote $\Phi_{\hat{w}_0^{j-1}, \Sigma}^j$, where $\hat{w}_0^{j-1} = \{\bar{w}^0, \bar{w}^1, \bar{w}^2, \dots, \bar{w}^{j-1}\}$, for brevity.

Lemma 4.3. *The renormalization microscope maps a set Υ^i onto an element of partition \mathcal{Q}_{jn} for Σ .*

Proof. The claim holds for $j = 1$ by the definition (25) of the elements of the partition.

Assume that it $\Phi_{\hat{w}_0^j, \Sigma}^j(\Upsilon^i)$ is an element of partition \mathcal{Q}_{jn} for Σ .

Consider $\Phi_{\hat{w}_0^j, \Sigma}^{j+1}(\Upsilon^i)$:

$$\Phi_{\hat{w}_0^j, \Sigma}^{j+1}(\Upsilon^i) = \Psi_{\bar{w}^0}^{\Sigma} \circ \Psi_{\bar{w}^1}^{\mathcal{R}\Sigma} \circ \dots \circ \Psi_{\bar{w}^j}^{\mathcal{R}^j\Sigma}(\Upsilon^i).$$

By assumption,

$$\Phi_{\hat{w}_1^j, \mathcal{R}\Sigma}^j(\Upsilon^i) \equiv \Psi_{\bar{w}^1}^{\mathcal{R}\Sigma} \circ \dots \circ \Psi_{\bar{w}^j}^{\mathcal{R}^j\Sigma}(\Upsilon^i)$$

is an element of the partition of level jn for the pair $\mathcal{R}\Sigma$, that is, by (25)

$$\Phi_{\hat{w}_1^j, \mathcal{R}\Sigma}^j(\Upsilon^i) = (\mathcal{R}\Sigma)^{\bar{v}} \circ L_{\mathcal{R}\Sigma} \circ L_{\mathcal{R}^2\Sigma} \circ \dots \circ L_{\mathcal{R}^j\Sigma}(\Upsilon^i),$$

for some admissible $\bar{v} = (\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m)$. Therefore, using the shorthand

$$\mathcal{R}\Sigma = (A_1, B_1),$$

we have:

$$\begin{aligned}
\Phi_{\hat{w}_0^j, \Sigma}^{j+1}(\Upsilon^i) &= \Psi_{\bar{w}^0}^\Sigma \circ \Phi_{\hat{w}_1^j, \mathcal{R}\Sigma}^j(\Upsilon^i), \\
&= \Sigma^{\bar{w}^0} \circ L_\Sigma \circ (\mathcal{R}\Sigma)^{\bar{v}} \circ L_{\mathcal{R}\Sigma} \circ \dots \circ L_{\mathcal{R}^j \Sigma}(\Upsilon^i) \\
&= \Sigma^{\bar{w}^0} \circ L_\Sigma \circ (B_1^{\beta_m} \circ A_1^{\alpha_m} \circ \dots \circ B_1^{\beta_0} \circ A_1^{\alpha_0}) \circ L_{\mathcal{R}\Sigma} \circ \dots \circ L_{\mathcal{R}^j \Sigma}(\Upsilon^i) \\
&= \Sigma^{\bar{w}^0} \circ L_\Sigma \circ \Lambda_\Sigma^{-1} \circ H_\Sigma \circ \left(\left(\Sigma^{\tilde{t}_n} \right)^{\beta_m} \circ \left(\Sigma^{\tilde{s}_n} \right)^{\alpha_m} \circ \dots \circ \left(\Sigma^{\tilde{t}_n} \right)^{\beta_0} \circ \left(\Sigma^{\tilde{s}_n} \right)^{\alpha_0} \right) \circ \\
&\quad \circ H_\Sigma^{-1} \circ \Lambda_\Sigma \circ L_{\mathcal{R}\Sigma} \circ \dots \circ L_{\mathcal{R}^j \Sigma}(\Upsilon^i) \\
&= \Sigma^{\bar{w}^0} \circ \left(\Sigma^{\tilde{t}_n} \right)^{\beta_m} \circ \left(\Sigma^{\tilde{s}_n} \right)^{\alpha_m} \circ \dots \circ \left(\Sigma^{\tilde{t}_n} \right)^{\beta_0} \circ \left(\Sigma^{\tilde{s}_n} \right)^{\alpha_0} \circ L_\Sigma \circ \dots \circ L_{\mathcal{R}^j \Sigma}(\Upsilon^i) \\
&= \Sigma^{\bar{u}} \circ L_\Sigma \circ \dots \circ L_{\mathcal{R}^j \Sigma}(\Upsilon^i),
\end{aligned}$$

for some index \bar{u} . By (25), the latter is an element of the partition $\mathcal{Q}_{(j+1)n}$. \square

Since $\mathcal{R}^l \Sigma$ converges to ζ_λ at a geometric rate, the function $\Psi_{\bar{w}}^{\mathcal{R}^l \Sigma}$ converges to the function $(\psi_{\bar{w}}^{\zeta_\lambda}, \psi_{\bar{w}}^{\zeta_\lambda^*})$, defined in Proposition 2.1, at a geometric rate in C^1 -metric. Therefore, by Proposition 2.1, there exists a neighborhood \mathcal{S} in $W_{\text{loc}}^s(\zeta_\lambda)$, and a sufficiently large $n = km$, such that

$$\|D\Psi_{\bar{w}}^{\mathcal{R}^l \Sigma}|_{\Upsilon^i}\|_\infty < \frac{1}{2},$$

whenever $\mathcal{R}^l \Sigma \in \mathcal{S}$.

For every $\Sigma \in W_{\text{loc}}^s(\zeta_\lambda)$, there exists $i_0 \in \mathbb{N}$ such that $\mathcal{R}^i \Sigma \in \mathcal{S}$ for $i \geq i_0$. Hence, there exists $C = C(\Sigma)$, such that

$$\|D\Phi_{\hat{w}, \Sigma}^j|_{\Upsilon^i}\|_\infty < \frac{C}{2^j}, \quad (26)$$

and thus the renormalization microscope is a uniform metric contraction.

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Let

$$\mathcal{R}^r(H_{l_*, \mu}) \equiv \Sigma = (A, B) \in W^s(\zeta_\lambda)$$

for some $r \in \mathbb{N}$.

Select a distinct point $(x_{\bar{w}}, y_{\bar{w}})$ in each of the sets $Q_{\bar{w}}^i \in \mathcal{Q}_{ln}$. Consider the ln -th dynamical partition \mathcal{P}_{ln} for the pair T_* as defined in Section 2. Consider a piecewise-constant map φ_l sending the element of the partition with a multi-index \bar{w} to $(x_{\bar{w}}, y_{\bar{w}})$. According to (26), the diameters of the sets $Q_{\bar{w}}^i$ decrease at a geometric rate. Thus, the maps φ_l converge uniformly to a continuous map φ of the interval $[-1, \rho_*]$ which is a homeomorphism onto the image. Set

$$\varphi([-1, \rho_*]) \equiv \gamma.$$

By construction,

$$\varphi \circ T_* = \Sigma \circ \varphi.$$

Let $\gamma_1 \subset K^+(H_{l_*,\mu})$ be the preimage of γ under renormalization rescaling, and set

$$\gamma_* \equiv \cup_{n \in \mathbb{N}} H_{l_*,\mu}(\gamma_1).$$

The conjugacy φ induces a conjugacy

$$\varphi_* : \mathbb{T} \rightarrow \gamma_*$$

between a rigid rotation and $H_{l_*,\mu}|_{\gamma_*}$. Hence, setting $l_* = e^{2\pi i \theta_1}$, we have

$$G^r(\theta_1) = \theta$$

for some $r \geq 0$.

Finally, since the limiting pair ζ_λ has a critical point at $z = 0$, the conjugacies φ and φ_* cannot be C^1 -smooth. Indeed, assume the contrary. This would imply that there exists $K > 1$ such that for every arc $J \subset \gamma_*$ and every $n \in \mathbb{N}$, we have

$$\frac{1}{K} \text{diam}(J) < \text{diam}(H_{l_*,\mu}^n(J)) < K \text{diam}(J). \quad (27)$$

However, let Ω_n, Γ_n denote the domains of the pair $p\mathcal{R}^n H_{l_*,\mu}$. Let $z \in \gamma_* \cap \Omega_n$ and $z' = H_{l_*,\mu}^{q_n}(z)$, and denote J_n the smaller subarc of γ_* bounded by these two points. Since

$$\mathcal{R}H_{l_*,\mu} \approx \zeta_\lambda$$

for large values of n , we have

$$\text{diam}(H_{l_*,\mu}^{q_{n+1}}(J_n)) \sim (\text{diam}(J_n))^2.$$

This clearly contradicts (27). \square

4.3. The curve γ_* bounds a Siegel disk. Let us define a ϱ -vertical cone field in the tangent bundle $T\Omega$ where Ω is a subdomain of \mathbb{C}^2 as

$$C_{(x,y)}^{v,\varrho} = \{(u,v) \in T_{(x,y)}\Omega, |u| < \varrho|v|\}.$$

Let $f : U \rightarrow \mathbb{C}$ be a holomorphic map. We consider two-dimensional perturbations of this map $F : \Omega \rightarrow \mathbb{C}^2$ of the form

$$F(x,y) = (w(x,y), h(x,y)) = (f(x) + \tau(x,y), g(x) + \chi(x,y)). \quad (28)$$

We note:

Proposition 4.4. *Suppose $|f'(x)| > \kappa$ and $|g'(x)| < K$ on the domain U for some $\kappa > 0$. Let F^{-1} be defined on $\Delta = F(\Omega)$.*

Then there exist $\epsilon > 0$ and $\varrho > 0$ such that the following holds. Suppose the uniform norms of τ and χ in (28) on Ω are bounded by ϵ . Given $\hat{\Delta} \Subset \Delta$, for every $(x,y) \in \hat{\Delta}$, denoting $(x_1, y_1) = F(x,y)$, we have

$$DF^{-1}|_{(x_1,y_1)} \left(C_{(x_1,y_1)}^{v,\varrho} \right) \subset C_{(x,y)}^{v,\varrho},$$

and $\|DF^{-1}\| > O\left(\frac{\kappa}{K\epsilon}\right)$ in $C^{v,\varrho}$.

Proof. Let $w_i(x, y)$ denote the i -th component of $w(x, y)$, $i = 1, 2$. Similarly for $h_i(x, y)$. A simple computation shows that:

$$DF^{-1}(x_1, y_1) \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{D(x, y)} \begin{bmatrix} h_2(x, y) - w_2(x, y) \\ -h_1(x, y) - w_1(x, y) \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{D(x, y)} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}, \quad (29)$$

where $D(x, y) = w_1(x, y)h_2(x, y) - w_2(x, y)h_1(x, y)$, and

$$\begin{aligned} |\tilde{u}| &< C\epsilon(|u| + |v|) < C\epsilon(\varrho + 1)|v|, \\ |\tilde{v}| &> (\kappa - C\epsilon)|v| - (|g_1(x_1)| + C\epsilon)|u| > (\kappa - C(1 + \varrho)\epsilon - \varrho K)|v|, \end{aligned}$$

and $|\tilde{u}| < \varrho|\tilde{v}|$ if $\varrho(\kappa - \varrho K) > C\epsilon(\varrho + 1)^2$. Furthermore, $|D(x, y)| < 2(K + C\epsilon)C\epsilon$ for some $C > 0$ and all $(x, y) \in \hat{\Delta}$. The lower bound on the operator norm $\|DF^{-1}\|$ on the vertical cone field follows. \square

As before, for $H_{l_*, \mu} \in W^s(\zeta_\lambda)$, we let Ω_n, Γ_n be the domains of the pair

$$Z_n \equiv p\mathcal{R}^n H_{l_*, \mu}.$$

For brevity, let us also write

$$\Delta_n \equiv \Omega_n \cup \Gamma_n \text{ and } \Delta'_n \equiv Z_n(\Delta_n).$$

Let α_* be the scaling factor α_n (see definition (6)) for the pair ζ_λ .

Proposition 4.5. *There exist $\delta > 0$, $k, N \in \mathbb{N}$ and $\varrho > 0$ such that the following holds. Let $|\mu| < \delta$ and $H_{l_*, \mu} \in W^s(\zeta_\lambda)$. Then for all $n \geq N$ the derivatives of the inverse branches of the restriction of the pair Z_n to the domains $\Delta_n \setminus \Delta_{n+k}$ preserve the vertical cone field $C^{v,\varrho}$ and expand vectors in $C^{v,\varrho}$ at a rate $O(|\alpha_*|^{-n-k}\delta^{-2^n})$.*

Proof. Let $Z_n = (f_n(x) + \tau_n(x, y), g_n(x) + \chi_n(x, y))$. By Lemma 3.1, the uniform norms of τ_n and χ_n on Δ_n are bounded from above by $O(\delta^{2^n})$.

Notice that Δ_{n+k} is an image of Δ_n under a linear map which converges to $(\alpha_*^k, 0)$ as $n \rightarrow \infty$. Therefore, $\text{dist}\{\partial(\Delta_{n+k} \cap \{y = 0\}), \{0\}\} > C|\alpha_*|^{-n-k}$, which gives $|f'_n(x)| > O(|\alpha_*|^{-n-k})$. The result follows from Proposition 4.4 with $\epsilon = O(\delta^{2^n})$ and $\kappa = O(|\alpha_*|^{-n-k})$. \square

The following result will be used in the proof of Proposition 4.7.

Lemma 4.6. (Löwner [Löw]) *Let $f : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic with $f(0) = 0$. If f extends to a homeomorphism of $\partial\mathbb{D}$, then f is a rotation.*

We can now complete the proof of Theorem C:

Proposition 4.7. *There exists $\delta > 0$ such that the following holds. Let $H_{l_*,\mu} \in W^s(\zeta_\lambda)$ with $|\mu| < \delta$ and let γ_* be the invariant curve constructed in Proposition 4.2. Then γ_* bounds a Siegel disk for $H_{l_*,\mu}$. The eigenvalue l_* is equal to λ_1 ,*

$$\lambda_1 = e^{2\pi i \theta_1} \text{ with } \theta = G^m(\theta_1) \text{ for some } m \geq 0. \quad (30)$$

Finally, there exists $\epsilon_1 > 0$ such that for all $|\mu| < \epsilon_1$ and for all λ_1 satisfying (30), we have $H_{\lambda_1,\mu} \in W^s(\zeta_\lambda)$.

Proof. Let us select k , N , and ϱ as in Proposition 4.5. Let $p \geq N$. Fix an open subdomain $\hat{\Delta}_p \Subset \Delta_p \cap \Delta'_p$. Since $H_{l_*,\mu}$ is a δ -small perturbation of the Siegel quadratic polynomial P_{λ_1} , we can select $\delta > 0$ small enough so that the map $H_{l_*,\mu}$ is normally hyperbolic in a sufficiently large neighborhood of the α -fixed point of P_{λ_1} . In particular, by Proposition 4.5, it is normally hyperbolic in the set $\hat{\Delta}_p \setminus \Delta_{p+k}$. Let \mathbf{q} be the fixed point of $H_{l_*,\mu}$ which is closest to the α -fixed point of P_{λ_1} . By the Graph Transform, the map $H_{l_*,\mu}$ has a weak stable/unstable/center manifold W of \mathbf{q} which is δ -close to the slice $\{y = 0\}$ (see [HPS]), and therefore $W \cap \hat{\Delta}_p \neq \emptyset$ if δ is sufficiently small.

Let us begin with the case when \mathbf{q} is attracting. By Proposition 4.5 the inverse branches of Z_{p+mk} , $m \geq 0$ are normally hyperbolic in $\Delta_{p+mk} \setminus \Delta_{p+(m+1)k}$. Therefore, the weak attracting submanifold W intersects Δ_{p+mk} for all $m \in \mathbb{N}$. We conclude that the invariant curve γ_* lies in the closure of W . By Proposition 4.2, the restriction $H_{l_*,\mu}|_{\gamma_*}$ is a homeomorphism, but not a diffeomorphism, therefore γ_* can not lie in the interior of \overline{W} . Applying Löwners Lemma 4.6, we arrive to a contradiction.

Suppose \mathbf{q} is hyperbolic. Then $W = W^u(\mathbf{q})$, the unstable manifold of \mathbf{q} , and successive applications of Proposition 4.5 as above imply that W extends to the invariant curve γ_* , which is then its boundary. This, again, contradicts Löwners Lemma 4.6.

Finally, suppose that \mathbf{q} is semi-neutral (that is, the linear part of the Hénon map at \mathbf{q} has a neutral eigenvalue of absolute value 1 and a dissipative eigenvalue of absolute value smaller than 1). In this case $W = W^c(\mathbf{q})$: it is only smooth, and *a priori*, not uniquely defined. The restriction $H_{l_*,\mu}|_W$ is not necessarily holomorphic.

By density of the irrationals of bounded type in the circle, we can choose a sequence $H_{l_j,\mu}$ of maps whose neutral eigenvalue $l_j = e^{2\pi i \vartheta_j}$ for some angle $\vartheta_j \in \mathbb{R} \setminus \mathbb{Q}$ of bounded type, converging to $H_{l_*,\mu}$. By continuity of the renormalization operator, for every $M \in \mathbb{N}$, there exists $J = J(M)$, such that for all $j > J(M)$ $H_{l_j,\mu}$ is $p + Mk$ times renormalizable with the height of the renormalizations coinciding with those for the map $H_{l_*,\mu}$. The Siegel disk W_j of $H_{l_j,\mu}$ is an analytic submanifold of \mathbb{C}^2 . Applying Proposition 4.5 to the inverse branches of Z_{p+mk}^j , $0 \leq m \leq M$ of $H_{l_j,\mu}$, and using considerations of dominated splitting, we can extend W_j for large j to intersect each $\hat{\Delta}_{p+km}$, $0 \leq m \leq M$. The rotation numbers

of the orbits of points in $W_j \cap \hat{\Delta}_{p+km}$, whose continued fraction expansion is given by the renormalization heights, approach θ_1 . Since, the rotation number of the orbits of $H_{l_j, \mu}|_{W_j}$ is constant, $\vartheta_j \mapsto \theta_1$, $DH_{l_*, \mu}(\mathbf{q}) = \lim_{j \rightarrow \infty} DH_{l_j, \mu}(\mathbf{q}_j)$, and $l_* = e^{2\pi i \theta_1}$. Therefore, W is a Siegel disk for $H_{l_*, \mu}$, and $H_{l_*, \mu}|_W$ is holomorphic. By Proposition 4.5 the submanifold W intersects Δ_{p+mk} for all $m \in \mathbb{N}$, and, therefore γ_* lies in the closure of W . By Proposition 4.2, the restriction $H_{l_*, \mu}|_{\gamma_*}$ is a homeomorphism, but not a diffeomorphism, therefore γ_* cannot lie in W .

Conversely, let $\lambda_1 = e^{2\pi i \theta_1}$ satisfy (30). As shown in Theorem 4.1, if μ is small enough, then the family $l \mapsto H_{l, \mu}$ intersects the stable set of ζ_λ near P_{λ_1} . Denote $l = \lambda_2$ the parameter of the intersection. As we have shown above, if $|\mu| < \epsilon$, then $\lambda_2 = e^{2\pi i \theta_2}$, where $\theta = G^j(\theta_2)$. The digits in the continued fraction expansion of θ_2 correspond to the periods of renormalizations of $H_{\lambda_2, \mu}$. By considerations of continuity, if μ is small enough, then the digits in the continued fractions of θ_2 and θ_1 coincide, and hence, $\lambda_2 = \lambda_1$. \square

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